A BIVARIATE FAILURE TIME MODEL, WITH DEPENDENCE DUE TO SHOCKS AND MIXED EFFECT

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A two-component system is considered, which is subject to external and possibly fatal shocks. The lifetimes of both components are characterized by their hazard rates. Each shock can cause the immediate failure of one single or of both components. Otherwise, the hazard rate of each component is increased by a non fatal shock of a random amount, with possible dependence between the simultaneous increments of the two failure rates. An explicit formula for the joint distribution of the bivariate lifetime of the two components is provided. Some positive dependence properties of the bivariate lifetime are found out. The influence of the shock model parameters on the bivariate lifetime is studied.

Keywords: Bivariate non-homogeneous compound Poisson process, Hazard rate process, Aging properties, Positive dependence properties.

1. Introduction

This paper is devoted to the survival analysis of a system subject to competing failure modes within an external stressing environment. The external environment is assumed to stress the system at random and isolated times according to a random shock model. Such a model can represent external demands e.g., which put some stress on the system at their arrivals. The occurrence of shocks is classically modeled through a non-homogeneous Poisson process and components lifetimes are characterized by their failure

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rates. The shocks are simultaneous for both components and each shock can be simultaneously fatal to both components, which induces some dependence between the components. Also, a shock increases the failure rates of the surviving components of a random increment, with possible dependence between simultaneous increments. Finally, following [1], the probability for a shock to be fatal depends on the shock's arrival time, which induces a last kind of dependence. Based on this setting, the aim of the paper is the study of the bivariate lifetime of the two components.

The paper is organized as follows: the model is specified in Section 2. An explicit formula for the joint survival function of the bivariate lifetime is provided in Section 3, as well as some positive dependence property. The influence of the model parameters on the bivariate lifetime is also discussed. Numerical experiments finally illustrate the study in Section 4.

2. The model

Out of the stressing environment, the two components are assumed to be independent. The lifetime of each component is characterized by its intrinsic hazard rate $h_i(t)$, $i = 1, 2$, or by the corresponding cumulative hazard rate $H_i(t) = \int_0^t h_i(u)du$, $i = 1, 2$. Stresses due to the external environment arrive by shocks, independently of the system intrinsic deterioration. The shocks occur at time $T_1, T_2...$ according to a non-homogeneous Poisson process $(N_t)_{t \geq 0}$ with intensity $\lambda(t)$ and cumulative intensity $\Lambda(t) = \int_0^t \lambda(x)dx$. The nth shock at time T_n increases the hazard rates of both components of a random amount $V_n = \left(V_n^{(1)}, V_n^{(2)} \right)$, where the increments $(V_n)_{n \geq 1}$ are independent and identically distributed (i.i.d.) and independent of the shocks arrival times $(T_n)_{n \geq 1}$. The simultaneous increments $V_n^{(1)}$ and $V_n^{(2)}$ at time T_n may however be dependent. Furthermore, a shock can be fatal, and can possibly induce the immediate failure of one single or of both components. The fatality of a shock does not depend on the system intrinsic deterioration but depends on the shock's arrival time. The following notations are used:

- $p_{00}(T_n)$: probability that the shock at time T_n induces the simultaneous failure of both components,
- $p_{11}(T_n)$: probability that the shock at time T_n induces no failure at all among the two components,
- $p_{01}(T_n)$: probability that the shock at time T_n is fatal only for the first component,

- $p_{10}(T_n)$: probability that the shock at time T_n is fatal only for the second component,

with $\sum_{0 \leq i,j \leq 1} p_{ij}(\cdot) = 1$.

The common distribution of the i.i.d. random vectors $V_n = \left(V_n^{(1)}, V_n^{(2)}\right)$, $n \in \mathbb{N}^*$ is denoted by $\mu(dv_1, dv_2)$. When subscript n is unnecessary, we drop it and set $V = \left(V^{(1)}, V^{(2)}\right)$ to be a generic copy of $V_n = \left(V_n^{(1)}, V_n^{(2)}\right)$. For $j = 1, 2$, the distribution of $V^{(j)}$ is denoted by $\mu_j(dv_j)$.

We set $(A_t)_{t \geq 0} = \left(A_t^{(1)}, A_t^{(2)}\right)_{t \geq 0}$ to be the bivariate compound Poisson process defined by

$$A_t^{(i)} = \sum_{n=1}^{N_t} V_n^{(i)}, \ i = 1, 2, \qquad A_t = \left(\sum_{n=1}^{N_t} V_n^{(1)}, \sum_{n=1}^{N_t} V_n^{(2)}\right)$$

where $\sum_{n=1}^{0} \ldots = 0$ and $\mathcal{F} = \sigma(A_s, s \geq 0)$ is the σ-field generated by $(A_t)_{t \geq 0}$.

Providing that it is functioning up to time t, the conditional hazard rate of the ith $(i = 1, 2)$ component at time t given $\mathcal{F} = \sigma(A_s, s \geq 0)$ is

$$r_i(t) = h_i(t) + A_t^{(i)}.$$

Let τ_i, $i = 1, 2$ be the lifetime of the ith component without taking into account the possibility of fatal shocks and let ξ_i, $i = 1, 2$ be the time of the first fatal shock for the ith component. We have

$$\mathbb{E}(\mathbf{1}_{\{\tau_i > t\}}|\mathcal{F}) = e^{-H_i(t)} e^{-\int_0^t A_s^{(i)} ds} = e^{-H_i(t)} e^{-\sum_{k=1}^{N_t} V_k^{(i)}(t-T_k)}$$

$$= e^{-H_i(t)} e^{-\sum_{k=1}^{\infty} V_k^{(i)}(t-T_k)^+} \tag{1}$$

and

$$\mathbb{E}(\mathbf{1}_{\{\xi_i > t\}}|\mathcal{F}) = \prod_{k=1}^{N_t} q_i(T_k)$$

where $q_1(\cdot) = p_{11}(\cdot) + p_{10}(\cdot)$ and $q_2(\cdot) = p_{11}(\cdot) + p_{01}(\cdot)$.

Given $\mathcal{F}$, the random variables τ_1 and τ_2 are assumed to be conditionally independent one with the other, and conditionally independent of ξ_1 and ξ_2. However, as the two components may fail simultaneously at each shock, ξ_1 and ξ_2 are not conditionally independent given $\mathcal{F}$. Given $\mathcal{F}$, their

conditional joint survival function is

$$\mathbb{E}(\mathbf{1}_{\{\xi_1>s\}}\mathbf{1}_{\{\xi_2>t\}}|\mathcal{F}) = \begin{cases} \prod_{i=1}^{N_t} p_{11}(T_i) \prod_{i=N_t+1}^{N_s} q_1(T_i) & \text{if } s \geq t, \\ \prod_{i=1}^{N_s} p_{11}(T_i) \prod_{i=N_s+1}^{N_t} q_2(T_i) & \text{if } s < t \end{cases} \tag{2}$$

where $\prod_{k=1}^{0} ... = 1$.

We set $Y = (Y_1, Y_2)$ to be the lifetime of the two components. It is easy to see that $Y_i = \min(\tau_i, \xi_i)$ for $i = 1, 2$.

3. Theoretical results

In all the paper, the results are provided without proof, due to the reduced size of the paper.

Proposition 3.1. *The joint survival function of $Y = (Y_1, Y_2)$ is given by*

$$\overline{F}_Y(s,t)$$

$$= e^{-H_1(s)-H_2(t)-\Lambda(\max(s,t))} exp\left(\int_0^{\min(s,t)} \tilde{\mu}((s-w)^+, (t-w)^+)p_{11}(w)\lambda(w)dw \right.$$

$$\left. +\mathbf{1}_{\{t\leq s\}} \int_t^s \tilde{\mu}_1(s-w)q_1(w)\lambda(w)dw + \mathbf{1}_{\{t>s\}} \int_s^t \tilde{\mu}_2(t-w)q_2(w)\lambda(w)dw \right) \tag{3}$$

where $\tilde{\mu}$ stands for the Laplace transform of the bivariate distribution μ of V, with

$$\tilde{\mu}(x_1, x_2) = \iint_{\mathbb{R}_+^2} e^{-x_1 v_1 - x_2 v_2}\mu(dv_1, dv_2) = \mathbb{E}\left(e^{-x_1 V^{(1)} - x_2 V^{(2)}} \right) \text{ for all } x_1, x_2 \geq 0$$

and $\tilde{\mu}^{(i)}$ stands for the Laplace transform of the distribution $\mu^{(i)}$ of $V^{(i)}$, with

$$\tilde{\mu}_i(x_i) = \int_0^\infty e^{-x_i v_i}\mu_i(dv_i) = \mathbb{E}\left(e^{-x_i V^{(i)}} \right) \text{ for } i = 1, 2 \text{ and all } x_i \geq 0.$$

We recall that Y_2 is said right-tail-increasing in Y_1, written $RTI(Y_2|Y_1)$, as soon as

$$\mathbb{P}(Y_2 > x_2|Y_1 > x_1)$$

is non decreasing in x_1 for all $x_2 \geq 0$. Also, Y_2 is said left-tail-decreasing in Y_1, written $LTD(Y_2|Y_1)$, as soon as $\mathbb{P}(Y_2 \leq x_2|Y_1 \leq x_1)$ is non increasing

in x_1 for all $x_2 > 0$. Both $RTI(Y_2|Y_1)$ and $LTD(Y_2|Y_1)$ properties are positive dependence properties, which imply association and positive quadrant dependence of Y, see [5] for more details on these different notions.

Proposition 3.2. *Both properties $RTI(Y_2|Y_1)$ and $RTI(Y_1|Y_2)$ are true.*

However, as will be shown in the next section, it is possible that one margin is not left-tail-decreasing in another margin, that is $LTD(Y_2|Y_1)$ property is no always true.

We now study the influence of different parameters on the bivariate lifetime Y. With that aim, two similar systems are considered, with identical parameters except one. We add an upper bar to all quantities referring to the second system. (For instance, we use $\bar{\lambda}(w)$ for the second system). The next result shows that, as expected, the more frequent the shocks are, the smaller the bivariate lifetime Y is.

Proposition 3.3. *Let us consider two systems with the same parameters except from the intensity of the non-homogeneous Poisson process. Assume that $\lambda(w) \leq \bar{\lambda}(w)$ for all $w \geq 0$. Then, $\overline{Y}$ is smaller than Y in the sense of the Weak Hazard Rate order ($\overline{Y} \leq_{whr} Y$), which means that $\bar{F}_Y(\mathbf{x})/\bar{F}_{\overline{Y}}(\mathbf{x})$ is increasing in $\mathbf{x} = (x_1, x_2) \in \{\mathbf{y} : \bar{F}_{\overline{Y}}(\mathbf{y}) > 0\}$). As a consequence [4], we also have $\overline{Y} \leq_{UO} Y$ (Upper Orthant Order), namely $\overline{F}_{\overline{Y}}(\mathbf{x}) \leq \overline{F}_Y(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}_+^2$ and*

$$m_{\bar{Y}}(\mathbf{x}) = \mathbb{E}(\bar{Y} - \mathbf{x}|\bar{Y} > \mathbf{x}) \leq m_Y(\mathbf{x}) = \mathbb{E}(Y - \mathbf{x}|Y > \mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}_+^2$, where $m_{\bar{Y}}$ and m_Y are the multivariate mean residual lifetimes of $\bar{Y}$ and Y.

The next result shows that the smaller the fatality of shocks is, the larger the bivariate lifetime Y is.

Proposition 3.4. *Let $U(w)$ with distribution $\mathbb{P}(U(w) = (i, j)) = p_{i\,j}(w)$ for all $i, j \in \{0, 1\}$ and $\bar{U}(w)$ defined in the same way with respect to the family $(\bar{p}_{i,j}(w))_{i,j \in \{0,1\}}$. Let us assume that $U(w) \leq_{UO} \bar{U}(w)$ for all $w \geq 0$. Then, $Y \leq_{UO} \overline{Y}$.*

Finally, assume that V and $\overline{V}$ are marginally identically distributed. In that case, Y and $\overline{Y}$ also are identically distributed. The upper orthant order is then equivalent to the Positive Quadrant Dependence order (PQD), which compares the dependence between the margins. We recall that the PQD order implies the Laplace transform order [3] so that the Laplace transform
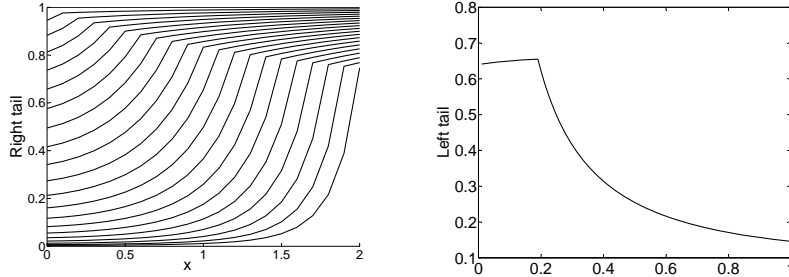
Figure 1. Example 4.1, RTI and not LTD property

order can also be considered as some comparison of the dependence between the margins. In that setting, the following result means that the more dependent the failure rates increments are (in the sense of the Laplace transform order), the larger the bivariate lifetime Y is.

Proposition 3.5. *Let V and $\overline{V}$ be marginally identically distributed. Assume V to be smaller than $\overline{V}$ in the sense of the bivariate Laplace transform order ($V \leq_{\mathcal{L}} \overline{V}$), namely $\mathbb{E}\left(e^{-s_1 V^{(1)} - s_2 V^{(2)}}\right) \leq \mathbb{E}\left(e^{-s_1 \overline{V}^{(1)} - s_2 \overline{V}^{(2)}}\right)$ for all $s_1, s_2 \geq 0$. Then $Y \leq_{PQD} \overline{Y}$.*

4. Numerical experiments

In all the following experiments, we take $h_i = 0$ for $i = 1, 2$. Also, we take $V^{(i)} = U^{(i)} + U^{(3)}$, $i = 1, 2$, where $U^{(1)}$, $U^{(2)}$ and $U^{(3)}$ are independent.

Example 4.1. The parameters are: $\lambda(x) = e^x$, $p_{11}(x) = p_{01}(x) = p_{10}(x) = \frac{e^{-x}}{4}$. Also, the $U^{(i)}$'s ($i = 1, 2, 3$) are exponentially distributed with respective parameters 1, 5 and 6. Figure 1 (left) shows the right tail $\frac{\overline{F}_Y(x, x_2)}{\overline{F}_{Y_1}(x)}$ with respect to x for various values of x_2. Whatever x_2 is, we observe that the right tail is always increasing so that, as expected, the $RTI\,(Y_2|Y_1)$ property is true. However, the left tail $\frac{F_Y(0.19, x)}{F_{Y_2}(x)}$ has been plotted in Figure 1 (right) and we observe that it is not monotonous. In that example, the $LTD\,(Y_2|Y_1)$ property is consequently false.

Example 4.2. The parameters are: $p_{11}(x) = p_{01}(x) = p_{10}(x) = \frac{e^{-x}}{4}$ and the $U^{(i)}$'s ($i = 1, 2, 3$) are gamma distributed with parameters $(a_i, 1)$ and $(a_1, a_2, a_3) = (1, 2, 3)$. We consider $\lambda(x) = x$ and $\overline{\lambda}(x) = 2x$, with $\lambda(x) \leq \overline{\lambda}(x)$. The quotient $r(\mathbf{x}) = \bar{F}_Y(\mathbf{x})/\bar{F}_{\overline{Y}}(\mathbf{x})$ is plotted in Figure 2,
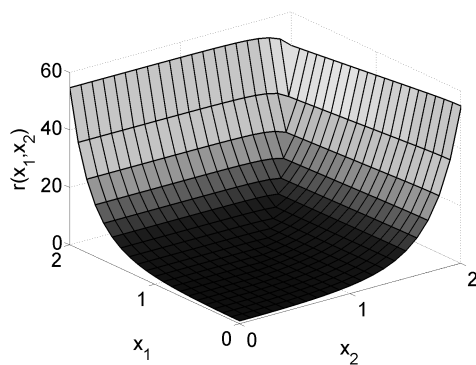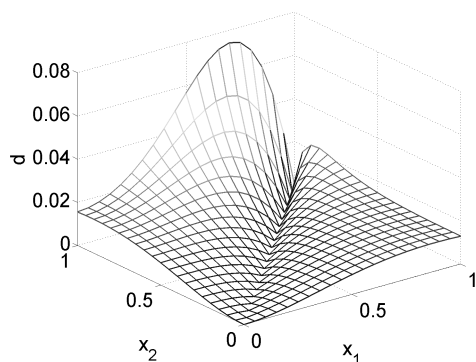
Figure 2. Example 4.2, $r(\mathbf{x})$ is monotone

Figure 3. Example 4.3, Influence of fatality of shocks

where Y and $\overline{Y}$ refer to $\lambda(x)$ and $\overline{\lambda}(x)$, respectively. We observe that, as expected, $r(\mathbf{x})$ is increasing in $\mathbf{x}$. It means $\overline{Y} \leq_{whr} Y$.

Example 4.3. This example shows numerically the influence of the fatality of shocks on lifetime. Let Y and $\overline{Y}$ be the lifetimes corresponding to $p = \left(p_{ij} = \frac{1}{4}\right)_{i,j=0,1}$ and $\bar{p} = \left(\bar{p}_{11} = \frac{1}{4}, \bar{p}_{01} = \frac{1}{6}, \bar{p}_{10} = \frac{1}{6}, \bar{p}_{00} = \frac{5}{12}\right)$. The $V^{(i)}$'s are the same as in Example 4.2. The other parameter is $\lambda(x) = 2x$. We have $\bar{p} \leq_{PQD} p$ and as expected, the difference $d = \overline{F}_Y - \overline{F}_{\overline{Y}}$ is always positive (see Figure 3). It means the bivariate lifetime Y is larger than $\overline{Y}$, in the sense of the positive quadrant dependence order.
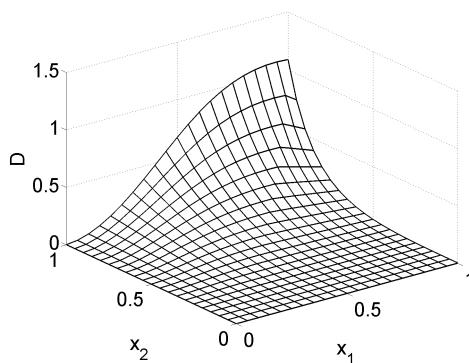
Figure 4.   Example 4.4, Influence of the dependence between $V^{(1)}$ and $V^{(2)}$

Example 4.4. This example illustrates the influence of the dependence between $V^{(1)}$ and $V^{(2)}$ on the bivariate lifetime. All parameters are similar as those in Example 4.2, except from the fact that, for the parameters of the gamma distributed $U^{(i)}$, $i = 1, 2, 3$, we take $(a_1, a_2, a_3) = (1, 2, 3)$ and $(\bar{a}_1, \bar{a}_2, \bar{a}_3) = (3, 4, 1)$. Then, $V^{(1)}$ and $\bar{V}^{(1)}$ (resp. $V^{(2)}$ and $\bar{V}^{(2)}$) are identically gamma distributed with parameter $(4, 1)$ (resp. $(5, 1)$). However, it is easy to check that $\overline{V} \leq_{\mathcal{L}} V$. We observe in Figure 4 that, as expected, the difference $D = \bar{F}_Y - \bar{F}_{\bar{Y}}$ is always positive, so that Y is larger than $\overline{Y}$, in the sense of the upper orthant order.

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